Correspondence

Weighted Least-Squares Design and Characterization of Complex FIR Filters

Amin G. Jaffer and William E. Jones

Abstract— **This correspondence presents two novel weighted least-squares methods for the design of complex coefficient finite impulse response (FIR) filters to attain specified arbitrary multiband magnitude and linear or arbitrary phase responses. These methods are computationally efficient, requiring only the solution of a Toeplitz system of N linear equations for an N-length filter that can be obtained in $o(N^2)$ operations. Illustrative filter design examples are presented.**

I. Introduction

The subject of real FIR filter design using both the weighted least-squares error (WLS) and Chebyshev criteria has been addressed extensively in the past [1]–[4]. More recently, the design of complex FIR filters that satisfy specified asymmetric amplitude or phase responses necessary in radar/sonar clutter suppression problems and other applications has been considered [5]–[9]. Nguyen [7] and Pei and Shyu [8] have employed the eigenfilter technique to approximately optimize the complex FIR filter WLS error design criterion. The eigenfilter technique, in addition to being only approximately optimal, requires the computation of a principal eigenvector by an iterative technique, where the number of iterations required for convergence can be quite large, resulting in heavy computational demands.

Two complex FIR filter WLS synthesis techniques—one for arbitrary phase response (unconstrained method) and the other incorporating the linear phase constraint (constrained method)—are developed here. The direct WLS optimization methods presented here utilize the complex gradient operator [10], which avoids decomposing the complex variables into real and imaginary parts. The linear-phase constrained method is developed using the complex Lagrange multiplier constraint, which is valid for either odd or even length filters. The filter coefficient vector is obtained very efficiently for both techniques as the solution of the resulting Hermitian–Toeplitz system of linear equations using a noniterative method (Levinson algorithm) [11]. Additionally, for a special but useful class of filters, our techniques result in a solution that altogether avoids the need for matrix inversion or the solution of a system of linear equations, thus reducing the computational demands significantly. The relationship between the constrained and unconstrained techniques is also examined. Finally, two illustrative filter design examples are presented with direct comparison of example two with the eigenfilter design example of Nguyen [7].

II. Weighted Least-Squares Complex FIR Filter Design

We derive here weighted least-squares algorithms for designing complex FIR filters to approximate arbitrary magnitude response constrained to have affine (generalized linear) phase as well as FIR filters with arbitrarily specified magnitude and phase responses. The conditions for complex FIR filters to posses affine phase are known in the literature and are also explicitly derived in [9]. Although the affine phase conditions are slightly more general, it suffices for our purposes to incorporate only the conjugate-symmetric constraints on the filter coefficients that generate linear phase as other filters of this class can be readily obtained from this form.

A. Constrained Weighted Least-Squares Technique

The conjugate-symmetric constraints are given by $h(n) = h^*(N-1-n)$, $n = 0, \cdots, N-1$, where $\underline{h} = [h(0), hzippy, \cdots, h(N-1)]^T$ represents the complex FIR filter coefficient vector.[1] We seek to obtain the coefficient vector $\underline{h}$ that minimizes the weighted integral squared-error criterion over the normalized frequency interval [0, 1)

$$J(\underline{h}) = \int_0^1 w(f)|z_D(f) - \underline{d}^H(f)\underline{h}|^2 df \tag{1}$$

subject to the above conjugate-symmetric constraints. Here, $w(f)$ is a nonnegative frequency weighting function, $z_D(f)$ is the desired complex frequency response, $\underline{d}(f)$ is the frequency "steering" vector given by $\underline{d}(f) = [1, e^{j2\pi f}, \cdots, e^{j2\pi(N-1)f}]^T$, the inner product $\underline{d}^H(f)\underline{h}$ represents the filter frequency response, and f represents the actual frequency normalized by the sampling frequency. The objective function given by (1) can accommodate arbitrary desired multiband magnitude responses including zero weighted frequency intervals.

The conjugate-symmetric constraints can be compactly represented by $\underline{h}^* = E\underline{h}$, where E is the $N \times N$ exchange matrix with ones on the cross diagonal and zeros elsewhere. Note that $E = E^T$, and $E^2 = I$, where I is the identity matrix. Incorporation of this vector constraint via the complex Lagrange vector $\underline{\lambda}$ formulation yields the augmented objective function

$$J_1(\underline{h}) = J(\underline{h}) - \underline{\lambda}^T[\underline{h}^* - E\underline{h}] - \underline{\lambda}^H[\underline{h} - E\underline{h}^*]. \tag{2}$$

Note that $J(\underline{h})$ and $J_1(\underline{h})$ are both real-valued functions for any complex vector $\underline{h}$. Expanding $J(\underline{h})$, differentiating with respect to $\underline{h}^*$ according to [10] (which treats a complex variable and its conjugate as independent variables) and equating to the null vector to satisfy the condition for the unique minimum yields

$$\frac{\partial J_1(\underline{h})}{\partial \underline{h}^*} = \int_0^1 \{ -w(f)z_D(f)\underline{d}(f) + w(f)\,\underline{d}(f)\underline{d}^H(f)\underline{h}\}df - \underline{\lambda} + E\underline{\lambda}^* = \underline{0}. \tag{3}$$

Let

$$Q = \int_0^1 w(f)\underline{d}(f)\underline{d}^H(f)df \tag{4}$$

$$\underline{u} = \int_0^1 w(f)z_D(f)\underline{d}(f)df. \tag{5}$$

Note that Q is a Hermitian–Toeplitz matrix that is fully defined by either its first row or column. Use of (4) and (5) in (3) yields

$$Q\underline{h} - \underline{u} = \underline{\lambda} - E\underline{\lambda}^*. \tag{6}$$

[1]The superscripts $*$, T, and H represent conjugate, transpose, and conjugate-transpose operations, respectively.

Manuscript received August 1, 1993; revised March 20, 1995. The associate editor coordinating the review of this paper and approving it for publication was Prof. Tamal Bose.

The authors are with Hughes Aircraft Company, Fullerton, CA 92634–3310 USA.

IEEE Log Number 9413863.

1053–587X/95$04.00 © 1995 IEEE

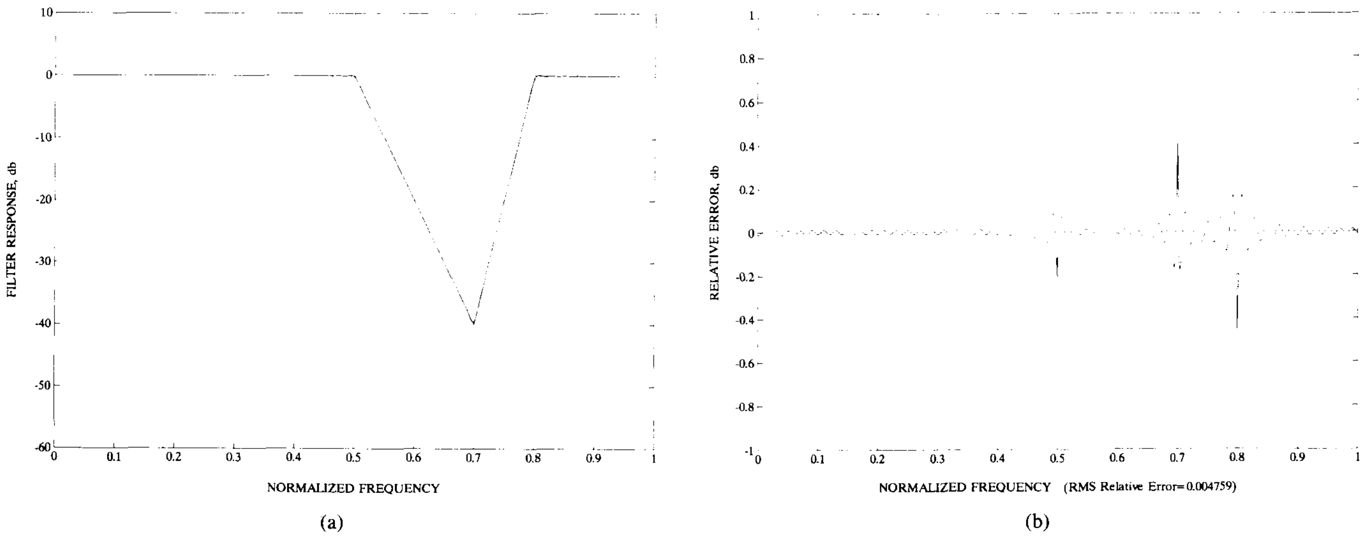

Fig. 1. Asymmetric v-notch filter design example: (a) Filter magnitude response; (b) relative error in magnitude response.

Let $\underline{\gamma} = \underline{\lambda} - E\underline{\lambda}^*$. Then

$$Q\underline{h} = \underline{u} + \underline{\gamma}. \tag{7}$$

We next determine $\underline{\gamma}$ so that the constraint $\underline{h}^* = E\underline{h}$ is satisfied. From (6) and (7), we have that

$$EQ\underline{h} = E\underline{u} + E\underline{\gamma} = E\underline{u} - \underline{\gamma}^*. \tag{8}$$

In addition, since Q is Hermitian-persymmetric, $EQ = Q^*E$ [11], and hence

$$Q^*E\underline{h} = E\underline{u} - \underline{\gamma}^*. \tag{9}$$

Subtracting (9) from the conjugate of (7) results in

$$Q^*[\underline{h}^* - E\underline{h}] = 2\underline{\gamma}^* + \underline{u}^* - E\underline{u}. \tag{10}$$

Applying the constraint $\underline{h}^* = E\underline{h}$ to (10) results in

$$\underline{\gamma} = \frac{1}{2}[E\underline{u}^* - \underline{u}]$$

and the solution for the filter coefficient vector as

$$Q\underline{h} = \frac{1}{2}[\underline{u} + E\underline{u}^*] \text{ or } \underline{h} = \frac{1}{2}Q^{-1}[\underline{u} + E\underline{u}^*]. \tag{11}$$

B. Unconstrained Weighted Least-Squares Technique

We derive here the unconstrained weighted least-squares complex FIR filter suitable for satisfying arbitrarily specified magnitude and phase responses (including nonlinear phase responses) that are necessary in many system applications. The solution immediately follows from the derivation in Section II-A by deleting the Lagrange multiplier constraints in (2), which results in $\underline{\gamma} = \underline{0}$ in (7), yielding the solution for the filter coefficient vector as

$$Q\underline{h} = \underline{u} \text{ or } \underline{h} = Q^{-1}\underline{u} \tag{12}$$

where Q and $\underline{u}$ are defined as before in (4) and (5).

C. Remarks

1) It can be readily verified that the constrained weighted least squares solution given by (11) does indeed satisfy the constraint $\underline{h}^* = E\underline{h}$, producing linear phase response, regardless of the desired complex response $z_D(f)$. The unconstrained solution given by (12) of course does not satisfy this property in general. However, it of interest to note that if the desired response is $z_D(f) = a_D(f)e^{j\phi_D(f)}$, where $a_D(f)$ is the desired magnitude response and $\phi_D(f)$ is linear phase with delay $\tau = (N-1)/2$, then the two solutions become one and the same. This can be seen by substituting $z_D(f) = a_D(f)e^{-j2\pi f(N-1)/2}$ in the expression for $\underline{u}$ in (5) and simplifying, resulting in the nth element of $\underline{u}$ being given by
$$[\underline{u}]_n = \int_0^1 w(f)a_D(f)e^{j2\pi f(n-(N+1)/2)}df.$$
It can also be shown that the nth element of $E\underline{u}^*$ is given by the same expression, whereupon $E\underline{u}^* = \underline{u}$ and (11) becomes $\underline{h} = Q^{-1}\underline{u}$, which is the same as (12).
2) Since the matrix Q is Hermitian–Toeplitz and, hence, fully defined by either its first row or column, the solution for the filter coefficient vector can be obtained quickly and accurately in $o(N^2)$ operations by the Levinson recursion algorithm [11] as opposed to general matrix inversion methods, which require $o(N^3)$ operations. Furthermore, our methods obtain the *true* WLS solution, whereas the eigenfilter method [5], [7] obtains an approximate WLS solution that requires a variable number of iterations to compute the principal eigenvector (depending on the eigenvalue spread) and that necessitates $o(N^2)$ operations per iteration. Note also that as a special but useful case, the Q matrix in this correspondence reduces to a scalar multiple of the identity matrix when the weighting function is uniform and the desired amplitude response encompasses the entire frequency interval without unspecified frequency bands, allowing the solution of the coefficient vector to be obtained trivially.
3) Since the constrained technique results in a conjugate-symmetric filter, it would ostensibly appear computationally attractive to obtain the solution directly in terms of half the coefficient vector for even length filters. However, the resulting solution is actually more demanding computationally than the one presented here due to the more complicated and non-Toeplitz structure of the associated matrix of the system of linear equations (see also [12]).

III. Illustrative Filter Design Examples

In this section, we examine two filter design examples that illustrate the use of the constrained and unconstrained WLS techniques presented here. A linear-phase asymmetric v-notch filter design example suitable for radar/sonar clutter suppression applications is used to

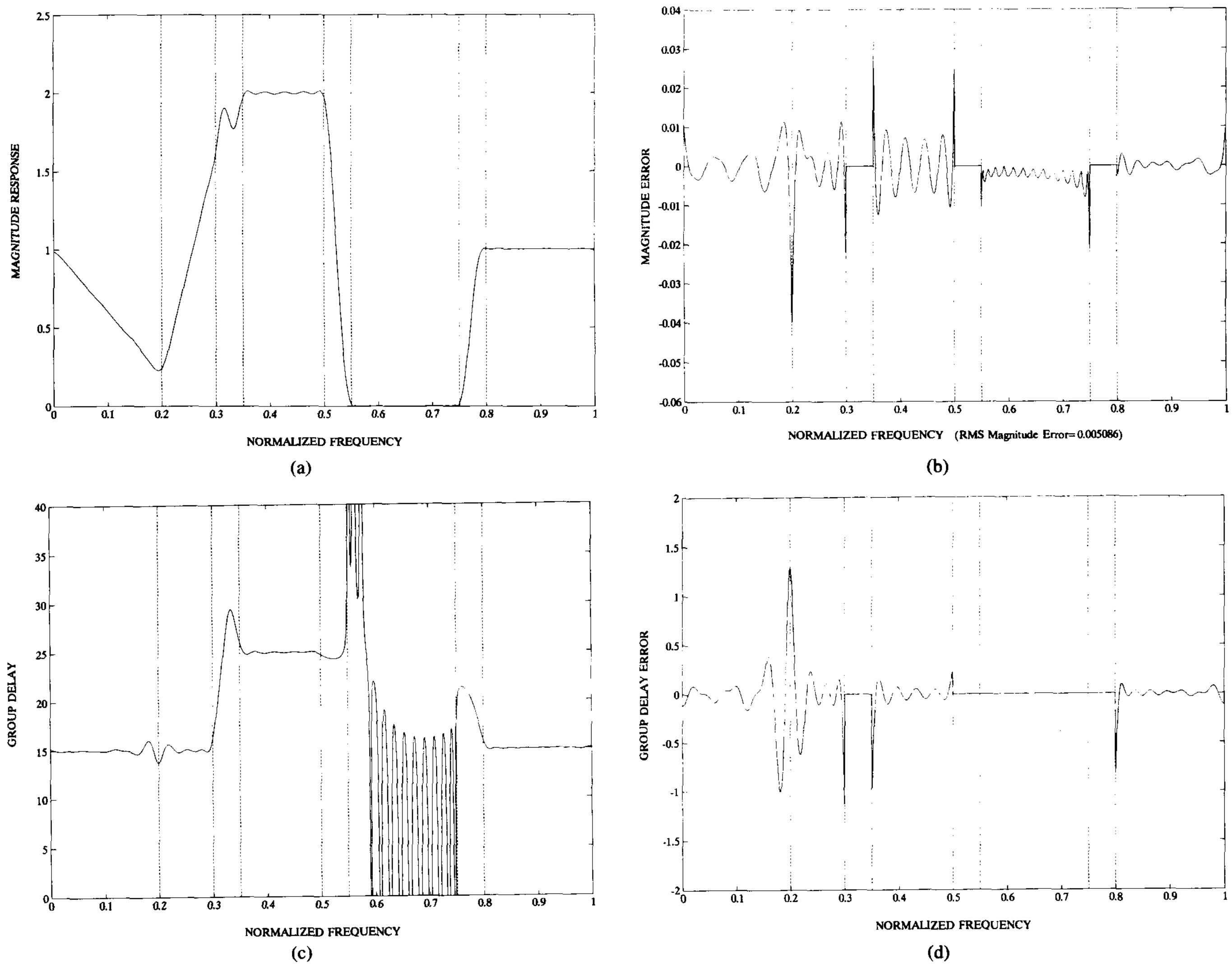

Fig. 2. Arbitrary transfer function example of Nguyen [7] using the unconstrained WLS method of this correspondence: (a) Magnitude response; (b) error in magnitude response; (c) group delay; (d) error in group delay.

illustrate the use of the constrained technique, whereas a direct comparison with the results of Nguyen [7] for his arbitrary transfer function filter design example is used to illustrate the use of the unconstrained technique.

The techniques developed here necessitate the evaluation of certain integrals for the computation of Q and $\underline{u}$ given by (4) and (5). In general, these integrals would require numerical integration; however, for an important subclass of practical filter design problems (including all of the examples presented here), these integrals are readily evaluated in closed form. In particular, integrals arising from filter design problems specified by multisegment piecewise linear and exponential amplitude (linear in log-amplitude) response specifications with uniform or inverse squared-error weighting can be evaluated in closed form, resulting in improved numerical efficiency and accuracy.

The Linear Phase Asymmetric V-Notch Filter Design Problem: The linear phase asymmetric v-notch filter design example is specified by the desired amplitude response function

$$|z_D(f)| = \begin{cases} 0 \text{ dB}, & 0 \leq f \prec 0.5 \\ [0, -40] \text{ dB}, & 0.5 \leq f \prec 0.7 \\ [-40, 0] \text{ dB}, & 0.7 \leq f \prec 0.8 \\ 0 \text{ dB}, & 0.8 \leq f \prec 1.0 \end{cases}$$

where the quantities in brackets specify the amplitudes at the endpoints of the exponential curve segment (linear in log-amplitude) that specifies the desired amplitude response in the frequency interval specified. The exponential amplitude response function is given by

$$|z_D(f)| = e^{\alpha_k + \beta_k f}, \text{ for } f \in [F_{k_1}, F_{k_2})$$

for the kth frequency interval $[F_{k_1}, F_{k_2})$ of the filter design specification. As the asymmetric v-notch filter has a 40-dB variation in its amplitude response, a minimum relative squared error estimation criterion is employed to balance the filter fit error amongst the specified frequency intervals evenly, resulting in the specification of the weighting function as

$$w(f) = \frac{1}{|z_D(f)|^2}.$$

A full derivation of the Q matrix and $\underline{u}$ vector for the relative squared error weighting and the linear and exponential amplitude response models is given in [9]. As this filter's amplitude response is asymmetric about any point in the normalized frequency domain (0 to 1 Hz.), it can only be generated with a complex FIR filter design technique; there is no purely real representation for these filter coefficients.

The amplitude response obtained by use of the constrained algorithm for the 101-tap complex linear phase FIR filter is given in Fig. 1(a), and the relative squared error, which is expressed in decibels, is given in Fig. 1(b). The constrained algorithm achieves a peak relative error of 0.41 dB at the frequency interval edges and a root-mean-square (RMS) error of 0.004759. The use of a relative squared-error minimization criterion is evident in the evenness of the error ripples across the large range of the filter's amplitude response Fig. 1(a).

The Arbitrary Filter Transfer Function Design Example of Nguyen [7]: Example 5, taken from Nguyen [7], is used to compare the unconstrained WLS technique presented here with the eigenfilter method of [7]. Nguyen's example consists of a specification with four passbands and one stopband with specified amplitude and phase requirements that, due to its asymmetry, necessitates a complex FIR filter synthesis technique. The unspecified frequency intervals are unweighted and do not contribute to the total fit error. Nguyen's example is specified with an absolute squared-error optimization criterion rather than the relative squared-error criterion used previously. The amplitude response attained by the unconstrained WLS technique for a 50-tap FIR filter is given in Fig. 2(a), the amplitude error in Fig. 2(b), the group delay in Fig. 2(c), and the group delay error in Fig. 2(d). The corresponding RMS errors are also shown in the figures. While the results obtained here are nearly identical to those of Nguyen, they represent the true WLS solution, which is also computed much more efficiently than the eigenfilter technique of [7] (see also Remark 2).

References

[1] T. W. Parks and J. H. McClellan, "Chebyshev approximations for nonrecursive digital filters with linear phase," *IEEE Trans. Circuit Theory*, vol. CT-19, no. 2, pp. 189–194, Mar. 1972.

[2] L. R. Rabiner and B. Gold, *Theory and Applications of Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975.

[3] X. Chen and T. W. Parks, "Design of FIR filters in the complex domain," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, no. 2, pp. 144–153, Feb. 1987.

[4] Y. C. Lim, J. H. Lee, C. K. Chen, and R. H. Yang, "A weighted least squares algorithm for quasiequiripple FIR and IIR digital filter design," *IEEE Trans. Signal Processing*, vol. 40, no. 3, pp. 551–558, Mar. 1992.

[5] P. P. Vaidyanathan and T. Q. Nguyen, "Eigenfilters: A new approach to least-squares FIR filter design and applications including Nyquist filters," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 11–23, Jan. 1987.

[6] K. Preuss, "On the design of FIR filters by complex Chebyshev approximation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, no. 5, pp. 702–712, May 1989.

[7] T. Q. Nguyen, "The design of arbitrary FIR digital filters using the eigenfilter method," *IEEE Trans. Signal Processing*, vol. 41, no. 3, pp. 1128–1139, Mar. 1993.

[8] S. C. Pei and J. J. Shyu, "Complex eigenfilter design of arbitrary complex coefficient FIR digital filters," *IEEE Trans. Circuits Syst.*, vol. 40, no. 1, pp. 32–40, Jan. 1993.

[9] A. G. Jaffer and W. E. Jones, "Constrained least-squares design and characterization of affine phase complex FIR filters," in *Proc. 27th Ann. Asilomar Conf. Signals, Syst., Comput.*, Nov. 1993, pp. 685–691.

[10] D. H. Brandwood, "A complex gradient operator and its applications in adaptive array theory," *Proc. Inst. Elec. Eng.*, vol. 130, pts. F and H, no. 1, pp. 11–16, Feb. 1983.

[11] S. L. Marple, *Digital Spectral Analysis with Applications*. Englewood Cliffs, NJ: Prentice-Hall, 1987.

[12] A. G. Jaffer, W. E. Jones, and T. J. Abatzoglou, "Weighted least-squares design of linear-phase and arbitrary 2-D complex FIR filters," presented at the *1995 IEEE Int. Conf. Acoust., Speech, Signal Processing* (ICASSP-95), May 1995.

Constraints on the Cutoff Frequencies of Mth-Band Linear-Phase FIR Filters

James M. Nohrden and Truong Q. Nguyen

Abstract—**In this correspondence, constraints are derived for the cutoff frequencies of linear-phase FIR Mth-band filters such that the filters have good passband and stopband characteristics, i.e. ones that very closely approximate an ordinary (non Mth-band) filter designed using some optimal method. Constraints on lowpass filters are first considered, and the results are extended to multiband filters.**

I. Introduction

Mth-band filters have found numerous applications in recent years [2]–[4], [9], [11], [14], [15]. In signal processing, Mth-band filters are used in 1-D [15] and 2-D [2] perfect reconstruction filter banks, nonuniform sampling [4], interpolation filters [14], and intersymbol interference rejection [11]. Additionally, Mth-band filters have found applications in antenna array design [3]. Mth-band filters are commonly designed as lowpass filters with cut-off frequencies at π/M. This does not have to be the case. In fact, bandpass and multiband Mth-band filters may be designed using the constrained set of cut-off frequencies derived in this paper.

Fig. 1 shows the desired response of a lowpass filter where ω_p and ω_s are the passband and stopband cutoff frequencies, respectively. δ_p and δ_s are the corresponding errors. The center frequency ω_c of a lowpass filter is defined as

$$\omega_c = \frac{\omega_p + \omega_s}{2}. \tag{1}$$

Let $\hat{H}(z)$ denote the transfer function of an odd length linear-phase FIR filter

$$\hat{H}(z) = \sum_{n=0}^{N-1} \hat{h}(n) z^{-n}, \quad \hat{h}(n)\,\text{real} \tag{2}$$

and define a noncausal shifted version of $\hat{H}(z)$ as $H(z) = z^L \hat{H}(z)$, where $L = (N-1)/2$. $H(z)$ is more suitable for the analytical work in this correspondence, whereas $\hat{H}(z)$ is actually implemented.

Optimal design techniques exist to minimize the frequency domain error for linear-phase FIR filters. One such example is the Remez algorithm [5], which minimizes the maximum error and therefore has an equiripple frequency response. Another algorithm is the eigenfilter approach [13], which minimizes the least squares error.

Let us define a good Mth-band filter as one that has approximately the same passband and stopband error characteristics as a non-Mth-band optimal filter with the same specifications. In other words, a good Mth-band filter is an Mth-band filter that is very nearly an optimal filter.

Manuscript received July 31, 1994; revised April 7, 1995. Thos work was supported by the U.S. Army Space and Strategic Defense Command under Air Force contract No. F19628-90-C-0002. The associate editor coordinating the review of this paper and approving it for publication was Dr. Kamal Premaratne.

J. M. Nohrden is with Massachusetts Institute of Technology, Lincoln Laboratory, Lexington, MA 02173-9108.

T. Q. Nguyen is with the Department of Electrical and Computer Engineering, University of Wisconsin, Madison, Madison, WI 53706 USA.

IEEE Log Number 9413860.

1053-587X/95$04.00 © 1995 IEEE