Constrained Least-Squares Design and Characterization of Affine Phase Complex FIR Filters

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Abstract

In many signal processing applications, the need arises for the design of complex coefficient finite impulse response (FIR) filters to meet the specifications which cannot be approximated by real coefficient FIR filters. This paper presents a new technique for the design of complex FIR filters based on minimizing a weighted integral squared-error criterion subject to the constraint that the resulting filter response be affine phase (i.e., generalize linear phase). The technique makes use of the necessary and sufficient conditions for a causal complex FIR filter to possess affine phase which are explicitly derived here. The method is non-iterative and computationally efficient. Several illustrative filter design examples are presented with excellent results.

1. Introduction.

The subject of the design of finite impulse response filters has a long history, as evidenced by a partial list of publications [1]-[5]. For the most part, however, the previous publications have been concerned with the design of real coefficient FIR filters whose frequency response functions *H(f)* necessarily satisfy $H(f)=H^*(-f)$, where * denotes complex conjugate. For a significant class of sensor signal processing problems, however, the desired frequency response will not necessarily satisfy this condition, e.g. the design of asymmetric notch filters for clutter cancellation problems in airborne radar or moving-platform active sonar systems. These systems and, in general, systems where analytic signals are to be processed to yield filter responses not satisfying this condition, mandate the need for complex coefficient FIR filters.

Although several authors have addressed the design of FIR filters by complex Chebyshev approximations [6]-[8], their works were restricted to real coefficient filters and their methods do not generalize readily to the complex coefficient case. Preuss [9] addressed the design of complex FIR filters using the Chebyshev norm and presented some interesting examples. However, his method involved a heuristic modification of the Remez exchange algorithm resulting in an iterative procedure that is not guaranteed to converge to the optimal solution. Weighted least-squares techniques [10], [11] also seem to have been applied only to the real coefficient case and, unlike the methods of this paper, appear to require a dense frequency sampling grid to model the desired amplitude response.

This paper is concerned with the design of complex coefficient FIR filters to satisfy a specified multiband amplitude response, based on minimizing a weighted integral squared-error criterion subject to the constraint that the resulting filter response possesses *affine* phase (i.e. linear phase with an offset). The incorporation of the affine phase constraint leads to good filter design and, moreover, is often a requirement in many system applications. The minimization is carried out subject to appropriate constraints on the filter coefficients (e.g. conjugate-symmetry constraints) needed to satisfy the affine phase property. These constraints for *complex* FIR filters are explicitly derived here in general form. An important feature of the present work is the use of the piecewise linear or exponential models to specify the desired multiband amplitude response, leading to an efficient, closed-form evaluation of certain integrals required in the computation of the optimal filter coefficient vector. This avoids the need for solving a discretized problem using a dense frequency sampling grid for the desired amplitude response, with the attendant problems in the transition bands. The filter design method requires only the solution of a set of N simultaneous linear equations (where N is the filter length) with a Hermitian-Toeplitz coefficient matrix, which can be obtained in only $\mathcal{O}(N^2)$ computations using the efficient Levinson or Trench algorithms [13].

Some illustrative complex coefficient filter design examples are also presented, including the design of asymmetric notch filters required for clutter suppression, and bandpass differentiators, with excellent results in general and direct comparison to previously published results for the latter example. Further

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examples, and the design of unconstrained complex FIR filters, can be found in [14].

2. Conditions for causal complex-valued FIR filters to possess affine phase

The conditions for real-valued causal FIR filters to possess linear phase, including linear phase with an offset, more appropriately termed *affine* phase, are well known and have been derived in [4] and [5]. However, their methods do not directly extend to the complex coefficient FIR filter case. It also appears that the necessary and sufficient conditions for complex FIR filters to be affine phase (including strict linear phase) do not appear to have been previously derived or stated, although special cases have been used in the literature [9]. The filter response for a causal N length FIR filter with complex coefficients $h(0), h(1), \ldots, h(N-1)$ is given by

$$H(f) = \sum_{n=0}^{N-1} h(n)\, e^{-j2\pi fn} \tag{1}$$

where, in (1), the sampling interval is taken as equal to *1* second so that f represents normalized frequency. We deduce, in the following theorem, the necessary and sufficient conditions for $H(f)$ to have an affine phase function

$$\Phi(f) = -2\pi f\alpha + \beta \tag{2}$$

for some α and β. The proof does not require separate treatments of even and odd length filters and is more general and precise than previous ones pertaining to the real filter case [5]. In order to exclude filters with leading or trailing zero coefficients which effectively alter the length of the filter, we impose the conditions $h(0) \neq 0$, $h(N-1) \neq 0$.

The main results of this Section are contained in the following theorem and its corollaries.

Theorem: *The filter response of a causal complex coefficient FIR filter, with coefficients* $h(0), h(1), \ldots, h(N-1)$ *where* $h(0) \neq 0$, $h(N-1) \neq 0$, *is affine phase of the form (2) if and only if* $h(n) = e^{j2\beta} h^*(N-1-n), \quad n = 0, \ldots, N-1$

Furthermore, the delay term α *is necessarily given by* $\alpha = (N-1)/2$

Proof: We prove necessity, i.e. the "only if" part, first. The affine phase property implies that (1) can be rewritten in the form

$$H(f) = e^{j\beta} e^{-j2\pi f\alpha} \left\{ e^{-j\beta} \sum_{n=0}^{N-1} h(n)\, e^{-j2\pi f(n-\alpha)} \right\} \tag{3}$$

where the term inside the braces is purely real, i.e. equal to its conjugate. Hence, equating this term to its conjugate and letting $z \equiv e^{j2\pi f}$ results in, explicitly,

$$h(0)z^{\alpha} + h(1)z^{\alpha-1} + \ldots + h(N-1)z^{\alpha-N+1} = e^{j2\beta}\left[h^*(0)z^{-\alpha} + h^*(1)z^{1-\alpha} + \ldots + h^*(N-1)z^{N-1-\alpha} \right] \tag{4}$$

In (4), z and its powers constitute a set of complex exponential functions that are linearly independent [15]. Hence (4) can only be satisfied for all values of z by equating the coefficients of like powers of z on both sides of (4). Since the powers of z on the left and right sides of (4) are, in descending order, $\{\alpha, \alpha-1, \ldots, \alpha-N+1\}$ and $\{N-1-\alpha, N-2-\alpha, \ldots, -\alpha\}$ respectively and since $h(0) \neq 0$, $h(N-1) \neq 0$ by assumption, we must have that the highest and lowest powers in the former set fall within the range of the latter set, i.e.,

$$-\alpha \leq \alpha \leq N-1-\alpha \tag{5}$$

$$-\alpha \leq \alpha - N + 1 \leq N-1-\alpha \tag{6}$$

But then, it follows from (5) that $0 \leq \alpha \leq (N-1)/2$ and from (6) that $(N-1)/2 \leq \alpha \leq N-1$, from which it follows that

$$\alpha = (N-1)/2 \tag{7}$$

precisely. Substituting (7) into (4), letting $m = N-1-n$ on the right side of (4) and rearranging yields

$$\sum_{n=0}^{N-1} h(n)\, z^{-(n-(N-1)/2)} = e^{j2\beta}\left\{ \sum_{n=0}^{N-1} h^*(N-1-n)\, z^{-(n-(N-1)/2)} \right\} \tag{8}$$

Equating coefficients of same powers of z on both sides of (8) (because of linear independence of the complex exponentials of z and its powers) yields

$$h(n) = e^{j2\beta} h^*(N-1-n), \quad n = 0, \ldots, N-1 \tag{9}$$

which, together with (7), proves the necessity part of the theorem. It is noted that setting $\beta = 0$ or $\pm\pi/2$ in (9) yields the conjugate symmetric and anti-conjugate symmetric filters respectively:

$$h(n) = h^*(N-1-n), \quad n = 0, \ldots, N-1 \tag{10}$$

$$h(n) = -h^*(N-1-n), \quad n = 0, \ldots, N-1 \tag{11}$$

Sufficiency: This follows immediately; for a proof see [14].

Corollary 1: If the filter coefficients are restricted to be real, it can be shown that the conditions (9) result in just two different constraints

$$h(n) = h(N-1-n), \qquad n = 0, \ldots, N-1 \tag{12}$$

$$h(n) = -h(N-1-n), \qquad n = 0,\ldots,N-1 \qquad (13)$$

which define the symmetric and anti-symmetric filters respectively, in agreement with [5]. The proof is omitted here for lack of space but can be found in [14].

Corollary 2: We will show that it suffices for filter design applications to apply the theorem for $\beta = 0$ and rotate the output phase by β. In (3), let $h_0(n) = e^{-j\beta}\, h(n)$ where the $h(n)$ satisfy the conditions of the theorem. For $n = 0,\ldots, N-1$, we have

$$h(n) = e^{j2\beta}\, h^*(N-1-n) \qquad (14)$$

and

$$h_0(n) = e^{j\beta}\, h^*(N-1-n) \qquad (15)$$
$$= h_0^*(N-1-n)$$

Hence $h_0(n)$ satisfies the conditions of the theorem for strict linear phase (with $\beta = 0$). Hence the filter response (3) can be written as

$$H(f) = e^{j\beta}\, H_0(f) \qquad (16)$$

where

$$H_0(f) = e^{-j2\pi\alpha}\left\{\sum_{n=0}^{N-1} h_0(n)\, e^{-j2\pi f(n-\alpha)}\right\} \qquad (17)$$

with $\alpha = (N-1)/2$. $H_0(f)$, as given by (17), is the frequency response of a strict linear phase filter and hence (16) states that the frequency response of an arbitrary affine phase filter can be obtained by multiplying $H_0(f)$ by $e^{j\beta}$, as was to be shown.

3. Linear phase FIR filter design by constrained weighted least-squares method.

We consider here the problem of designing complex coefficient FIR filters to approximate a specified complex valued frequency response by employing a weighted integral least-squares error criterion. Let $z_D(f)$ be the desired complex valued frequency response and $W(f)$ be a real nonnegative piecewise continuous weighting function. $z_D(f)$ is of the form $z_D(f) = a_D(f)\, e^{j\Phi_D(f)}$ where $a_D(f)$ and $\Phi_D(f)$ are the desired amplitude and phase responses respectively. Let $\underline{h} = [\, h(0), h(1), \ldots, h(N-1)\,]^T$ represent the complex coefficient FIR filter of length *N*. *H(f)*, as given by (1) can be written as

$$H(f) = \underline{d}^H(f)\,\underline{h} \qquad (18)$$

where $\underline{d}(f) = \left[\, 1, e^{j2\pi f}, \ldots, e^{j2\pi(N-1)f}\,\right]^T$

and the superscripts T and H denote the transpose and conjugate transpose operators. We seek to obtain the coefficient vector $\underline{h}$ that minimizes the weighted integral squared-error criterion

$$J(\underline{h}) = \int_0^1 W(f)\left|\, z_D(f) - \underline{d}^H(f)\,\underline{h}\,\right|^2 df \qquad (19)$$

subject to the constraint (1). It is noted that the criterion (19) allows for $z_D(f)$ being specified over compact subsets of the normalized frequency interval $[0,1)$ and that $W(f)$ may equal zero over some of the subsets. In the design examples given in Section 5, $W(f)$ is chosen so that criterion (19) becomes one of minimizing the *relative* integral squared error. This is discussed more fully in Section 4.

The conjugate symmetry constraint (10) on the filter coefficients can be compactly represented as

$$\underline{h}^* = E\,\underline{h} \qquad (20)$$

where E is the $N \times N$ exchange matrix with ones on the cross-diagonal and zeros elsewhere. The exchange matrix has the properties that $E^T = E$ and $E^2 = I$ (the identity matrix) so that $E^{-1} = E$. Note also that the formulation (20) applies to both even and odd length filters, so that separate treatments of these two cases are unnecessary.

The cost function (19) is a real-valued, nonnegative function of the complex vector $\underline{h}$. Its minimization may be accomplished by the use of a complex gradient operator and its associated matrix-vector calculus operations as described by Brandwood [13]. Let $\underline{h} = \underline{h}_x + j\,\underline{h}_y$ where $\underline{h}_x$ and $\underline{h}_y$ are the real and imaginary components of the complex vector $\underline{h}$. Define the complex gradient operator as

$$\nabla_{\underline{h}} = 1/2\left(\partial/\partial\underline{h}_x - j\,\partial/\partial\underline{h}_y\right) \qquad (21)$$

Then a necessary and sufficient condition for a stationary point of $J(\underline{h})$ is that $\nabla_{\underline{h}}\, J(\underline{h}) = \underline{0}$. Equation (19) can be written as

$$J(\underline{h}) = \int_0^1 W(f)\begin{bmatrix} |z_D(f)|^2 - z_D^*(f)\,\underline{d}^H(f)\,\underline{h} \\ -\underline{h}^H\,\underline{d}(f)\,z_D(f) + \underline{h}^H\,\underline{d}(f)\,\underline{d}^H(f)\,\underline{h}\end{bmatrix} df \qquad (22)$$

Using the constraint equation $\underline{h}^* = E\,\underline{h}$ or, equivalently, $\underline{h}^H = \underline{h}^T E$, we get

$$J(\underline{h}) = \int_0^1 W(f)\begin{bmatrix} |z_D(f)|^2 - z_D^*(f)\,\underline{d}^H(f)\,\underline{h} \\ -\underline{h}^T E\,\underline{d}(f)\,z_D(f) \\ +\underline{h}^T E\,\underline{d}(f)\,\underline{d}^H(f)\,\underline{h}\end{bmatrix} df \qquad (23)$$

Differentiating with respect to $\underline{h}$ (see [13] for details on applying the complex gradient operator to linear and quadratic forms), and equating to $\underline{0}$ yields

$$\nabla_{\underline{h}}\, J(\underline{h}) = \int_0^1 W(f)\begin{bmatrix} -z_D^*(f)\,\underline{d}^*(f) - E\,\underline{d}(f)\,z_D(f) \\ +\left(R + R^T\right)\underline{h}\end{bmatrix} df = \underline{0} \qquad (24)$$

where $R = E\,\underline{d}(f)\,\underline{d}^H(f)$. Rearrangement of (24) yields

$$\left[\int_0^1 W(f)\left(R+R^T\right)df\right]\underline{h} = \int_0^1 W(f)\left[E\,z_D(f)\,\underline{d}(f) + z_D^*(f)\,\underline{d}^*(f)\right]df \quad (25)$$

Although (25) expresses the filter coefficient vector $\underline{h}$ as the solution of a set of simultaneous linear equations, it can be simplified further to yield a more compact and computationally efficient form. First note that since $R = E\,\underline{d}(f)\,\underline{d}^H(f)$ and pre-multiplication of a matrix by E reverses its rows, we can readily show that $R = R^T$. Now let

$$Q = \int_0^1 W(f)\,\underline{d}(f)\,\underline{d}^H(f)\,df \quad (26)$$

$$\underline{r} = \int_0^1 W(f)\left[E\,z_D(f)\,\underline{d}(f) + z_D^*(f)\,\underline{d}^*(f)\right]df \quad (27)$$

Then (25) reduces to

$$Q\,\underline{h} = 1/2\,E\,\underline{r} \quad (28)$$

The Hermitian matrix Q is also Toeplitz since its $(m,n)th$ element, given by $\int_0^1 W(f)\,e^{j2\pi(m-n)f}\,df$, depends only on the difference $(m-n)$. Equation (28) represents a system of N simultaneous linear equations, with a Hermitian-Toeplitz coefficient matrix, for the solution of the filter coefficient vector $\underline{h}$. Additionally, since the Vandermonde type vectors $\underline{d}(f)$ are linearly independent for distinct values of f in the interval $[0,1)$, the matrix Q will be of full rank N (and positive-definite) as long as the range of the integration in (28) encompasses any interval or at least N discrete distinct points in the frequency domain where $W(f)$ is not zero. This will be true for any non-trivial, well-posed filter design problem, resulting in a unique solution for $\underline{h}$. The matrix Q is completely specified by either its first row or column and the system of linear equations (28) can be efficiently solved in $O(N^2)$ operations by the Levinson recursion or Trench algorithm [12], resulting in a significant computational savings over general matrix inversion techniques which require $O(N^3)$ operations.

Note that $\underline{h}$, as given by the solution of (28) actually satisfies the conjugate-symmetry constraint regardless of the specification of the desired phase response $\Phi_D(f)$, which need not be linear. However, since the constraint (20) would only be imposed for linear phase filter design problems, it would be appropriate for the desired phase response $\Phi_D(f)$ to be specified as linear phase with a delay of $(N-1)/2$, i.e. as $\Phi_D(f) = -2\pi f(N-1)/2$.

The Q matrix and the $\underline{r}$ vector, required in (28) are specified as follows: The first column of the Hermitian-Toeplitz matrix Q, which completely defines Q, is given by

$$[Q]_{m,1} = \int_0^1 W(f)\,e^{j2\pi(m-1)f}\,df, \qquad m = 1,\ldots,N \quad (29)$$

Using $\Phi_D(f) = -2\pi f(N-1)/2$, the nth element of the $\underline{r}$ vector specified by (27) can be shown to be given by

$$[\underline{r}]_n = 2\int_0^1 W(f)\,a_D(f)\,e^{-j2\pi f(n-(N-1)/2)}\,df \quad (30)$$

In the filter design examples presented in Section 5 the weighting function corresponds to the squared *relative* error. Furthermore, since most filter design problems, including the examples in Section 5, require approximating the frequency response over multiple subbands, which may be disjoint, the weighting function specializes to

$$W(f) = \frac{c_k}{a_{Dk}^2(f)}, \qquad F_{k1} \le f \le F_{k2}, \quad k = 1,\ldots,M \quad (31)$$

where M is the number of frequency subbands of interest, $a_{Dk}(f)$ is the desired amplitude response over the kth subband and c_k are additional discrete weights included in (31) to permit emphasizing certain frequency segments over others. Under these assumptions, (29) and (30) reduce to

$$[Q]_{m,1} = \sum_{k=1}^{M} c_k \int_{F_{k1}}^{F_{k2}} \frac{e^{j2\pi f(m-1)}}{a_{Dk}^2(f)}\,df, \quad m = 1,\ldots,N \quad (32)$$

$$[\underline{r}]_n = 2\sum_{k=1}^{M} c_k \int_{F_{k1}}^{F_{k2}} \frac{e^{-j2\pi f(n-(N-1)/2)}}{a_{Dk}(f)}\,df, \quad n = 1,\ldots,N \quad (33)$$

Two particular types of amplitude response functions $a_{Dk}(f)$, the linear amplitude and the linear log-amplitude or exponential response models, are employed in the examples of Section 5. These result in the closed-form expressions for the integrals in (32) and (33) which are readily evaluated and also yield excellent performance as demonstrated in Section 5.

4. Weighting and amplitude response models.

In this section two specific amplitude response functions, the linear and exponential, used in conjunction with relative square-error weighting, are used as models for the linear phase complex FIR filter design technique, the constrained technique, defined in Section 4. The linear model is defined by

$$a_{Dk}(f) = \alpha_k + \beta_k\,f \quad (34)$$

$$\text{where} \quad \beta_k = \left(\frac{A_{k2}-A_{k1}}{F_{k2}-F_{k1}}\right)$$

$$\text{and} \quad \alpha_k = A_{k1} - F_{k1}\,\beta_k$$

and the exponential model by

$$a_{Dk}(f) = e^{\gamma_k + \delta_k f} \quad (35)$$

$$where \quad \delta_k = \left(\frac{\ln(A_{k2}) - \ln(A_{k1})}{F_{k2} - F_{k1}}\right)$$

$$and \quad \gamma_k = \ln(A_{k1}) - F_{k1}\,\delta_k$$

for the kth frequency segment.

The exponential model implies that the log-amplitude is linear over the frequency segment which is particularly suitable for many applications. The exponential model is also computationally more efficient than the linear model when they are used in conjunction with relative squared-error weighting. The Q matrix and $\underline{r}$ vector used in the constrained algorithm described in Section 3 are evaluated for the exponential model substituting (35) into the kth term of (32) and (33) to yield the expressions

$$[Q_k]_{m,1} = c_k \int_{F_{k1}}^{F_{k2}} \frac{e^{j2\pi f(m-1)}}{e^{2(\gamma_k+\delta_k f)}} df, \qquad m = 1,\ldots,N \quad (36)$$

$$[\underline{r}_k]_n = 2\,c_k \int_{F_{k1}}^{F_{k2}} \frac{e^{-j2\pi f(n-(N-1)/2)}}{e^{(\gamma_k+\delta_k f)}} df, \quad n = 1,\ldots,N \quad (37)$$

These expressions can then be solved in closed form as

$$[Q_k]_{m,1} = c_k\, e^{-2\gamma_k} \left. \frac{e^{2f(j\pi(m-1)-\delta_k)}}{2(j\pi(m-1)-\delta_k)} \right|_{f=F_{k1}}^{f=F_{k2}} \quad m = 1,\ldots,N \quad (38)$$

$$[\underline{r}_k]_n = -2c_k\, e^{-\gamma_k} \left. \frac{e^{-f(j2\pi(n-(N-1)/2)+\delta_k)}}{(j2\pi(n-(N-1)/2)+\delta_k)} \right|_{f=F_{k1}}^{f=F_{k2}} \quad n = 1,\ldots,N \quad (39)$$

Note that (36) and (37) reduce to simpler expressions when $m=1$ and $\delta_k = 0$ for (36), and when $n = (N+1)/2$ and $\delta_k = 0$ for (37).

The linear segment model used in conjunction with the relative squared-error weighting, for the constrained algorithm is determined by the Q matrix and $\underline{r}$ vector. These are formed by substituting (34) into (32) and (33) yielding the expressions

$$[Q_k]_{m,1} = c_k \int_{F_{k1}}^{F_{k2}} \frac{e^{j2\pi f(m-1)}}{(\alpha_k+\beta_k f)^2} df, \qquad m = 1,\ldots,N \quad (40)$$

$$[\underline{r}_k]_n = 2\,c_k \int_{F_{k1}}^{F_{k2}} \frac{e^{-j2\pi f(n-(N-1)/2)}}{(\alpha_k+\beta_k f)} df, \quad n = 1,\ldots,N \quad (41)$$

The general result of evaluating (40) and (41) is given by

$$[Q_k]_{m,1} = \frac{c_k}{\beta_k} e^{-j2\pi(m-1)\frac{\alpha_k}{\beta_k}} \quad (42)$$

$$\bullet \left[\left(j2\pi\frac{(m-1)}{\beta_k}\right) Ei\left(j2\pi(m-1)\frac{u}{\beta_k}\right) - \frac{1}{u} e^{j2\pi(m-1)\frac{u}{\beta_k}} \right]_{u=A_{k1}}^{u=A_{k2}}$$

$$[\underline{r}_k]_n = \frac{2c_k}{\beta_k} e^{j2\pi(n-(N-1)/2)\frac{\alpha_k}{\beta_k}} \quad (43)$$

$$\left[Ei\left(-j2\pi(n-(N-1)/2)\frac{u}{\beta_k}\right) \right]_{u=A_{k1}}^{u=A_{k2}}$$

where $Ei(x)$ is the exponential integral as defined in Gradshteyn and Ryzhik [16] and Abramowitz and Stegun [17]. It can be easily evaluated in terms of the sine and cosine integrals $si(x)$ and $ci(x)$ related by

$$Ei(\pm jx) = ci(x) \pm j\, si(x) \quad (44)$$

Note that (40) and (41) reduce to simpler expressions when $\beta_k = 0$ or $m=1$ for (40), and when $\beta_k = 0$ or $n = (N+1)/2$ for (41).

5. Illustrative filter design examples.

This section examines two applications of the constrained algorithm, the asymmetric v-notch filter and the bandpass differentiator examined by Preuss [9]. The examples in this section employ the amplitude response models of Section 4 used in conjunction with relative square-error weighting.

The filter design examples in this Section were generated using a MATLAB program based on the results of Sections 3 and 4. By exploiting the Toeplitz structure of matrix Q in as defined in Section 3, the Levinson or Trench algorithms [12] can be employed to reduce the computational complexity of the solution from an $\mathcal{O}(n^3)$ solution to an $\mathcal{O}(n^2)$ solution. Using these techniques the 101-tap V-Notch filter design problem below was generated in 3.35 seconds in MATLAB on an 486DX IBM-PC clone running at 50 MHz.

The Asymmetric V-Notch Filter Design Problem: The asymmetric V-Notch filter examined here is defined by the specification

$$|H_D(f)| = \begin{cases} 0 \;\; dB, & 0 \le f \prec .5 \\ [0,-40] \;\; dB, & .5 \le f \prec .7 \\ [-40,0] \;\; dB, & .7 \le f \prec .8 \\ 0 \;\; dB, & .8 \le f \prec 1.0 \end{cases} \quad (45)$$

where the quantities in brackets in (45) are the end-points of the *exponential* amplitude response function over the frequency interval specified ($[A_{k1}, A_{k2}]$ for the kth frequency interval in (34)). The asymmetry in the desired amplitude response can arise in moving platform radar or active sonar systems where the unwanted clutter returns exhibit Doppler shifts that are largely "down-Doppler" relative to the Doppler frequency of the moving platform. Due to this asymmetry, the filter cannot be synthesized by conventional real FIR filter design techniques as the coefficients can never be represented as a purely real sequence. For this example, each frequency subband is equally weighted in the sense that the c_k parameter defined in (31) is equal to *1* for each segment. Equations (28), (38) and (39) are used to evaluate the filter coefficients as functions of the parameters specified in the design specification.

Figure 1 illustrates a sample asymmetric v-notch filter generated with 101 coefficients. The filter coefficients are indeed conjugate-symmetric as expected from the arguments made in Section 2. The relative error of the filter synthesis

$$e_{rel,\,dB}(f) = 20.*\log\frac{|H_a(f)|}{|H_d(f)|} \tag{46}$$

is plotted in Figure 2. The RMS error used here is defined by

$$RMS = \sqrt{\sum_{m=1}^{M}\int_{F_{m1}}^{F_{m2}}\left|\frac{|H_a(f)|-|H_d(f)|}{|H_d(f)|}\right|^2 df} \tag{47}$$

The 101-tap linear phase V-notch filter achieves maximum relative error of about 0.47 dB and an rms error of .004759.

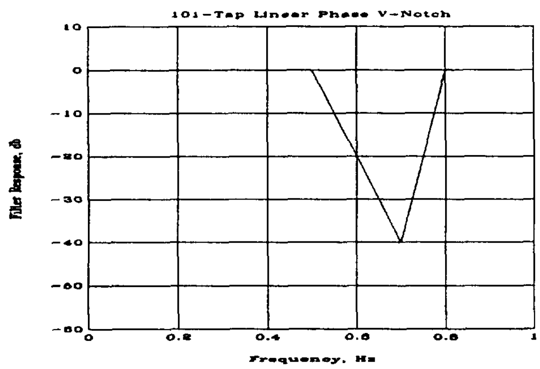

Figure 1. 101-Tap Asymmetric V-Notch Filter Amplitude Response.

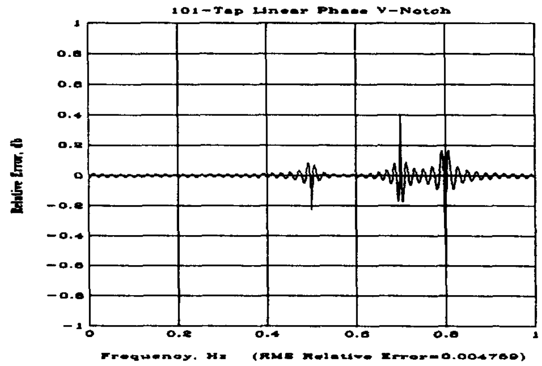

Figure 2. 101-Tap Asymmetric V-Notch Filter Error.

The Bandpass Differentiator: In this Section the linear phase bandpass differentiator filter specified by Preuss [9] is generated using the constrained technique as described in Sections 3 and 4. In Preuss' paper this filter is specified as

$$H_d(f) = \begin{cases} j2\pi f\, e^{-j2\pi 15.5f} & 0.03750 \le f \le 0.42500 \\ 0 & 0.57500 \le f \le 0.96250 \end{cases} \tag{48}$$

Note that this filter contains an affine phase offset produced by the *j* constant in (48). By the argument given in Section 2, it suffices to ignore this affine phase offset for the purposes of generating a linear phase filter and later multiply the complex FIR filter coefficients by this *j* term which forms the final filter. This post-multiplication doesn't change the filter's amplitude response.

The realization of the bandpass differentiator as specified in (48) using the constrained algorithm is given by

$$|H_d(f)| = \begin{cases} [.0710,.8700] & c_k = 2\times 10^6, & .0355 \le f \prec .4350 \\ [.8700,.0009] & c_k = 100, & .4350 \le f \prec .5650 \\ [.0009,.0009] & c_k = 1, & .5650 \le f \prec .9625 \end{cases}$$

where the quantities in brackets are the end-points of the *linear* amplitude response function over the frequency interval specified ($[A_{k1},A_{k2}]$ for the *k*th frequency interval in (35)). The quantities c_k are the auxiliary weights applied to the subbands as defined in (31). This realization was found to yield satisfactory fit errors in the passband while preserving acceptable stop-band rejection. It is also desirable to weight the stop-band to a lesser degree than the passband to prevent an inordinate amount of effort being employed in flattening the stop-band. The results of our realization of the bandpass differentiator can be seen in Figure 3.

The filter design fit error used in this example is specified by

$$e_{rel}(f) = \frac{|H_a(f)|-|H_d(f)|}{|H_d(f)|} \tag{49}$$

Preuss obtained a relative peak amplitude error of $\pm 2 \times 10^{-4}$ over the frequency interval $[.0375,.4250]$. The results of the application of the constrained algorithm are given in Figures 3 and 4. Note that the constrained algorithm achieves a smaller relative error over most of the passband (rms value of 1.084×10^{-4}), although the peak relative error is greater at about -4.3×10^{-4}. This result is consistent with the nature of the weighted integral squared error and the Chebyshev criteria.

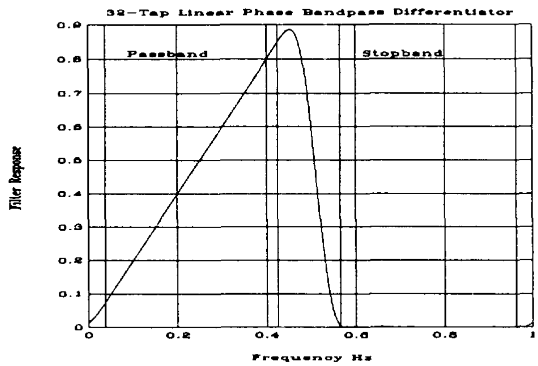

Fig 3. 32-Tap Bandpass Differentiator Amplitude Response.

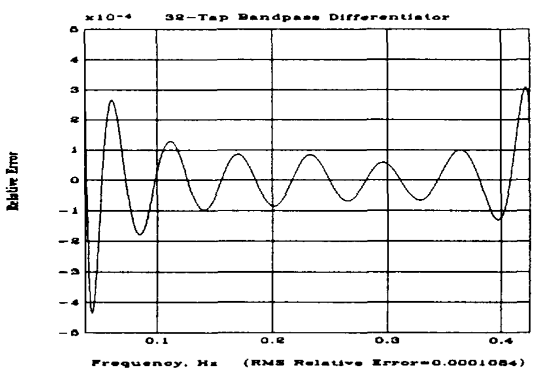

Fig 4. 32-Tap Bandpass Differentiator Filter Fit Error.

References

[1] L. R. Rabiner and R. W. Schafer, "Recursive and non-recursive realizations of digital filters designed by frequency sampling techniques," IEEE Trans. Audio Electroacoustics, vol. AU-19, no. 3, pp. 200-207, Sept. 1971.

[2] T. W. Parks and J. H. McClellan, "Chebyshev approximations for non-recursive digital filters with linear phase," IEEE Trans. Circuit Theory vol. CT-19, no. 2, pp. 189-194, March 1972.

[3] J. H. McClellan and Parks, "A unified approach to the design of optimal FIR linear phase digital filters," IEEE Trans. Circuit Theory, vol. CT-20, pp. 697-701, Nov. 1973.

[4] L. R. Rabiner and B. Gold, *Theory and Applications of Digital Signal Processing*, Prentice-Hall, Inc., New Jersey, 1975.

[5] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*. Edgewood Cliffs, New Jersey, Prentice-Hall, 1989.

[6] X. Chen and T. W. Parks, "Design of FIR filters in the complex domain," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-35, no. 2, pp. 144-153, Feb. 1987.

[7] N. N. Chit and J. S. Mason, "Complex Chebyshev approximation for FIR digital filters," IEEE Trans. Signal Processing, vol. 39, No. 1, pp. 49-54, Jan. 1991.

[8] A. S. Alkhairy, K. G. Christian and J. S. Lim, "Design and characterization of optimal FIR filters with arbitrary phase," IEEE Trans. Signal Processing, vol. 41, No. 2, pp. 559-572, Feb. 1993.

[9] K. Preuss, "On the design of FIR filters by complex Chebyshev approximation," IEEE Trans. Acoust., Speech and Signal Processing, vol. 37, No. 5, pp. 702-712, May 1989.

[10] Y. C. Lim, J. H. Lee, C. K. Chen and R. H. Yang, "A weighted least squares algorithm for quasi-equiripple FIR and IIR digital filter design," IEEE Trans. Signal Processing, vol. 40, No. 3, pp. 551-558, March 1992.

[11] C. S. Burrus, A. W. Soewito and R. A. Gopinath, "Least squared error FIR filter design with transition bands," IEEE Trans. Signal Processing, vol. 40, No. 6, pp. 1327-1340, June 1992.

[12] S. L. Marple, *Digital Spectral Analysis with Applications*. Englewood Cliffs, New Jersey, Prentice-Hall, 1987.

[13] D. H. Brandwood, "A complex gradient operator and its applications in adaptive array theory," IEE Proceedings, vol. 130, Pts. F and H, No. 1, pp. 11-16, February 1983.

[14] A. G. Jaffer and W. E. Jones, "Weighted Least-Squares Design and Characterization of Complex FIR Filters", submitted to IEEE Trans. Signal Processing.

[15] Tom M. Apostol, *Calculus, Volume II*. Second edition, John Wiley & Sons, New York, 1969.

[16] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. Orlando, Florida: Academic Press, 1980.

[17] M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions*. Washington, D.C.: National Bureau Of Standards Applied Mathematics Series #55, 1972.